Derivation of the wave equation from Maxwell's equation

We can for example consider the electric field and start with Faraday's law:

$$\nabla \times \boldsymbol{\mathcal{E}} = -\mu_0 \frac{\partial \boldsymbol{\mathcal{H}}}{\partial t}$$

$$\nabla \times \nabla \times \boldsymbol{\mathcal{E}} = \nabla \times \left(-\mu_0 \frac{\partial \boldsymbol{\mathcal{H}}}{\partial t}\right)$$

$$\nabla \times \nabla \times \boldsymbol{\mathcal{E}} = -\mu_0 \frac{\partial}{\partial t}(\nabla \times \boldsymbol{\mathcal{H}})$$

$$\nabla \times \nabla \times \boldsymbol{\mathcal{E}} = -\mu_0 \varepsilon_0 \frac{\partial^2 \boldsymbol{\mathcal{E}}}{\partial t^2}$$

We use the identity:

$$\nabla \times \nabla \times \boldsymbol{\mathcal{E}} = \nabla(\nabla \cdot \boldsymbol{\mathcal{E}}) - \nabla^2 \boldsymbol{\mathcal{E}}$$

We get:

$$-\nabla^2 \boldsymbol{\mathcal{E}} = -\mu_0 \varepsilon_0 \frac{\partial^2 \boldsymbol{\mathcal{E}}}{\partial t^2}$$

$$\nabla^2 \boldsymbol{\mathcal{E}} - \frac{1}{{c_0}^2}\frac{\partial^2 \boldsymbol{\mathcal{E}}}{\partial t^2} = 0$$

With $c_0 = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}$

Derivation of Helmhotz equation

We consider the complex representation $\mathbf{E}(\mathbf{r}, t) = \mathbf{E}(\mathbf{r}) e^{i\omega t}$

The wave equation simply reduces to:

$$\nabla^2 \mathbf{E}(\mathbf{r}, t) - \frac{1}{{c_0}^2}\frac{\partial^2 \mathbf{E}(\mathbf{r}) e^{i\omega t}}{\partial t^2} = 0$$

$$\nabla^2 \mathbf{E}(\mathbf{r}, t) + \frac{\omega^2}{{c_0}^2}\mathbf{E}(\mathbf{r}) e^{i\omega t} = 0$$

$$\nabla^2 \mathbf{E}(\mathbf{r}, t) + k^2 \mathbf{E}(\mathbf{r}, t) = 0$$

With $k = \frac{\omega}{c_0}$

Example of calculating the coherence of a wave.

Let's consider a purely monochromatic wave at some fixed point in space. It can be written: $\mathbf{U(}t\mathbf{)} = U_0 e^{i\omega_0 t}$. We here assume its phase is null, without loss of generality.

$$\langle \mathbf{U}^*(t)\mathbf{U}(t+\tau)\rangle = \frac{1}{T}\int_{t_0}^{t_0+T} U_0^* e^{-i\omega_0 t} U_0 e^{i\omega_0 (t+\tau)} dt$$

$$\langle \mathbf{U}^*(t)\mathbf{U}(t+\tau)\rangle = \frac{1}{T}\int_{t_0}^{t_0+T} |U_0|^2 e^{i\omega_0 \tau} dt$$

$$\langle \mathbf{U}^*(t)\mathbf{U}(t+\tau)\rangle = |U_0|^2 e^{i\omega_0 \tau}$$

Similarly,

$$\langle \mathbf{U}^*(t)\mathbf{U}(t+\tau)\rangle = |U_0|^2$$

We get that

$$g(\tau) = \frac{\langle \mathbf{U}^*(t)\mathbf{U}(t+\tau)\rangle}{\langle \mathbf{U}^*(t)\mathbf{U}(t)\rangle} = e^{i\omega_0 \tau}$$

$$|g(\tau)| = 1$$

A monochromatic wave is a perfectly coherent wave. However, as we will see, such wave does not exist, as all light exhibits a finite linewidth.